ELASTICITY CONSTANTS OF AXIAL TEXTURES IN THE VOIGT-REUSS-HILL APPROXIMATION

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Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 9, No. 2, pp. 48–50, 1968

Various expressions are used to describe the changes in the elastic properties of a polycrystalline material during the process of plastic deformation, annealing, etc. leading to the formation of texture. Thus, for axial texture of a material composed of cubic crystallites two groups of formulas have been proposed [1, 2]:

$$
\begin{aligned}
&C_{11}^T = C_{22}^T = \overline{C}_{11} + {}^3/_{20}\alpha C, && S_{11}^T = S_{22}^T = \overline{S}_{11} + {}^3/_{20}\,\alpha S;\\
&C_{33}^T = \overline{C}_{11} + {}^2/_5\,\alpha C, && S_{33}^T = \overline{S}_{11} + {}^2/_5\,\alpha S;\\
&C_{13}^T = C_{23}^T = \overline{C}_{12} - {}^1/_5\,\alpha C, && S_{13}^T = S_{23}^T = \overline{S}_{12} - {}^1/_5\,\alpha S;\\
&C_{12}^T = \overline{C}_{12} + {}^1/_{20}\,\alpha C, && S_{12}^T = \overline{S}_{12} + {}^1/_{20}\,\alpha S;\\
&C_{44}^T = C_{55}^T = \overline{C}_{44} - {}^1/_5\,\alpha C, && S_{44}^T = S_{55}^T = \overline{S}_{44} - {}^4/_5\,\alpha S;\\
&2C_{66}^T = C_{11}^T - C_{12}^T, && S_{66}^T = 2(S_{11}^T - S_{12}^T);\\
&K_V = K_0 && K_R = K_0 .
\end{aligned} \tag{1}
$$

Here, $\alpha = a_4/9$ characterizes the degree of order of the orientation of the crystallites in the texture, a_4 is the amplitude of the fourth term of the expansion of the orientation distribution function in spherical functions [1, 2]. The quantities $\overline{C}_{ij}$ and $\overline{S}_{ij}$ are the mean elasticity characteristics of the isotropic polycrystal, calculated in the Voigt and Reuss approximations, respectively, from the elasticity constants of the single crystal, C_{ij} and S_{ij}:

$$
\begin{aligned}
&\overline{C}_{11} = C_{11} - {}^2/_5\,C,\ \overline{C}_{12} = C_{12} + {}^1/_5\,C,\ \overline{C}_{44} = C_{44} + {}^1/_5\,C;\\
&\overline{S}_{11} = S_{11} - {}^2/_5\,S,\ \overline{S}_{12} = S_{12} + {}^1/_5\,S,\ \overline{S}_{44} = S_{44} + {}^4/_5\,S.
\end{aligned} \tag{2}
$$

Finally, the quantities C and S in (1) and (2) are the parameters of anisotropy for the elastic material:

$$C = C_{11} - C_{12} - 2C_{44},\ S = S_{11} - S_{12} - {}^1/_2\,S_{44}.$$

An analysis of the methods of evaluating expressions (1) shows that the formulas for C_{ij}^T and S_{ij}^T account for, only approximately, the change in the elastic properties of the texture and are essentially equivalent to the calculations for a quasi-isotropic polycrystal in the Voigt and Reuss approximations, respectively. This means that expressions (1) are not suitable for describing the properties of materials composed of strongly anisotropic crystals, since in this case $\overline{C}_{ij}$ and $\overline{S}_{ij}$ differ considerably from the experimental values of the constants of the quasi-isotropic material. Moreover, formulas (1) cannot be used to calculate the properties of strong textures. In these cases, a higher approximation is required, and formulas (1) give the necessary basis for calculating the elasticity constants in the Voigt-Reuss-Hill approximation [3].

In the case of a quasi-isotropic polycrystal, Voigt and Reuss averaging (2) gives the upper and lower limits, respectively, for the true values of the material constants. As an additional refinement, Hill [3] has proposed using the arithmetic or geometric mean of the Voigt and Reuss constants. Numerous experiments have shown that the arithmetic mean is in good agreement with the experimental data and differs little from the theoretical values of the polycrystal constants obtained by more exact methods [4]. Our problem is as follows: using (1), to calculate in this approximation the elasticity properties of axial texture. The final expressions prove to be simpler if the geometric means obtained from (1) are used in the computation.

In what follows, we denote the elasticity constants of the texture in the required approximation by C_{ij}^T and S_{ij}^T, and the elasticity constants of the quasi-isotropic polycrystal in the same approximation by C_{ij}^0 and S_{ij}^0.

Easiest to calculate are the shear moduli $C_{44}^* = (S_{44}^*)^{-1}$ and $C_{66}^* = (S_{66}^*)^{-1}$ of the texture. In fact,

$$
\begin{aligned}
&C_{44}^* = \left(\frac{\overline{C}_{44}}{\overline{S}_{44}}\right)^{1/2} = C_{44}^0\left(1 - \frac{\alpha}{5}\, f(a)\right);\quad C_{44}^0 = \left(\frac{\overline{C}_{44}}{\overline{S}_{44}}\right)^{1/2}\\
&a = \frac{C_{11} - C_{12}}{2C_{44}} = \frac{S_{44}}{2(S_{11} - S_{12})};\quad f(a) = \frac{25(a^2 - 1)}{(3 + 2a)(2 + 3a)}.
\end{aligned} \tag{3}
$$

Similarly, for C_{66}^* we have

$$C_{66}^* = (C_{66}/S_{66})^{1/2} = C_{44}^0\,[1 + {}^1/_{20}\,\alpha f(a)]. \tag{4}$$

The calculation of the remaining constants is more complicated: the quantities C_{11} and S_{11}, C_{12} and S_{12}, etc., are not reciprocal, and direct calculation of the geometric means is impossible. Expressions (1) determine the components of the matrices $\|C_{ij}\|$ and $\|S_{ij}\|$ in the Voigt and Reuss approximations, respectively. We will consider the components of the reciprocal matrices

$$\|C_{ij}^*\|\ \|S_{ij}^*\| = I \text{ (I is the unit matrix).}$$

By the usual methods [5] it is possible to find separately the eigenvalues λ_k of the matrix $\|C_{ij}\|$ and Λ_k of the matrix $\|S_{ij}\|$.

For a medium with hexagonal symmetry, only four of the six eigenvalues are different. The eigenvalues λ_k^* of the matrix $\|C_{ij}^*\|$ can be calculated from the rule of the geometric mean, i.e.,

$$\lambda_k^* = (\lambda_k/\Lambda_k)^{1/2} = (\Lambda_k^*)^{-1} \quad (k = 1, \ldots, 4) \tag{5}$$

where Λ_k^* are eigenvalues of the matrix $\|S_{ij}\|$.

Thus, we have four relations containing five unknown texture elasticity constants C_{ij}^* and S_{ij}^*. Two of these relations have already been used in (3) and (4). The missing relation for reconverting from λ_k^* to C_{ij}^* can be obtained from the condition

$$K_V^* = K_V = K_R = K_R^* = K_0 \tag{6}$$

which expresses the fact that the bulk modulus of a texture composed of cubic crystals is independent of both the method of averaging and the degree of order of the crystal orientations. Using (5) and (6), we can find the elasticity constants for axial texture:

$$
\begin{aligned}
&S_{11}^* = S_{11}^0 - {}^3/_{40}\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{11}^* = C_{11}^0 + {}^3/_{40}\,\alpha f(a)\,(C_{11}^0 - C_{12}^0);\\
&S_{33}^* = S_{11}^0 - {}^1/_5\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{12}^* = C_{11}^0 + {}^1/_{40}\,\alpha f(a)\,(C_{11}^0 - C_{12}^0);\\
&S_{13}^* = S_{12}^0 + {}^1/_{10}\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{33}^* = C_{11}^0 + {}^1/_5\,\alpha f(a)\,(C_{11}^0 - C_{12}^0);\\
&S_{12}^* = S_{12}^0 - {}^1/_{40}\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{13}^* = C_{12}^0 - {}^1/_{10}\,\alpha f(a)\,(C_{11}^0 - C_{12}^0);\\
&S_{44}^* = S_{44}^0 - {}^2/_5\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{44}^* = C_{44}^0 - {}^1/_{10}\,\alpha f(a)\,(C_{11}^0 - C_{12}^0);\\
&S_{66}^* = S_{44}^0 - {}^1/_{10}\,\alpha f(a)\,(S_{11}^0 - S_{12}^0);\\
&C_{66}^* = C_{44}^0 + {}^1/_{40}\,\alpha f(a)\,(C_{11}^0 - C_{12}^0).
\end{aligned} \tag{7}
$$

From these expressions we find the anisotropy parameters

$$
\begin{aligned}
&S^* = S_{33}^* - S_{13}^* - {}^1/_2\,S_{44}^* = -{}^1/_2\,\alpha f(a)\,(S_{11}^0 - S_{12}^0)\\
&C^* = C_{33}^* - C_{13}^* - 2C_{44}^* = {}^1/_2\,\alpha f(a)\,(C_{11}^0 - C_{12}^0).
\end{aligned} \tag{8}
$$

Replacing α in expression (7) with its value from (8), we arrive at formulas analogous to those obtained previously [2] using Reuss averaging:

$$S_{11}^* = S_{11}^0 + {}^3/_{20}\, S^*; \quad C_{11}^* = C_{11}^0 + {}^3/_{20}\, C^*;$$

$$S_{33}^* = S_{11}^0 + {}^2/_5\, S^*; \quad C_{33}^* = C_{11}^0 + {}^2/_5\, C^*;$$

$$S_{13}^* = S_{12}^0 - {}^1/_5\, S^*; \quad C_{13}^* = C_{12}^0 - {}^1/_5\, C^*;$$

$$S_{12}^* = S_{12}^0 + {}^1/_{20}\, S^*; \quad C_{12}^* = C_{12}^0 + {}^1/_{20}\, C^*;$$

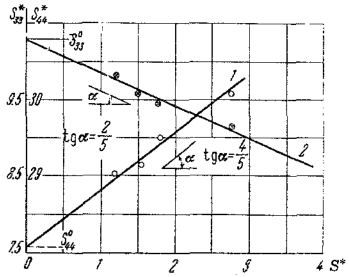
$$S_{44}^0 = S_{44}^0 - {}^4/_5\, S^*; \quad C_{44}^* = C_{44}^0 - {}^1/_5\, C^*;$$

$$S_{66}^* = S_{44}^0 + {}^1/_5\, S^*; \quad C_{66}^* = C_{44}^0 + {}^1/_{20}\, C^*. \tag{9}$$

We note that, in form, expressions (9) are identical with (1), but the quantities S_{ij}^0 and C_{ij}^0, S_{ij}^* and C_{ij}^* are components of the reciprocal matrices of the quasi-isotropic aggregate and texture. Since expressions (1) and (9), which determine the properties of the texture in different approximations, are identical in form, it may be assumed that they also retain the same form in higher approximations. This means that from measurements of the quantities S_{33}^*, S_{13}^*, and S_{44}^* on textured material it is possible to find exact values of the quantities S_{11}^0 and S_{44}^0 for the quasi-isotropic aggregate. The quantities S_{33}^*, S_{13}^*, and S_{44}^* can be measured using the dynamic resonance method [2, 8].

In order to verify the expressions obtained (9), we conducted an experiment on textured cylindrical specimens of L-62 brass. These were fabricated as follows: the brass blank was annealed at 600° C for 1 hr and then subjected to plastic deformation by drawing. As a result of this treatment we obtained axial textures of varying intensity. The results of the measurements are shown graphically in the figure, which also gives the theoretical slope of the lines based on Eqs. (9). Clearly, these lines have the same slope as the experimental lines and their points of intersection with the ordinate axis give the values of the shear modulus ($1/S_{44}^0$) and the Young's modulus ($1/S_{33}^0$) of the quasi-isotropic aggregate.

Thus, in order to determine the elasticity constants of a quasi-isotropic material on textured specimens it is sufficient to measure a single specimen. In particular, the Young's and shear moduli of the



Lines 1 and 2 give the variation of S_{44}^* and S_{33}^*, respectively, with the anisotropy parameter of the texture S^T. The theoretical slope of the lines from relations (9) is given for comparison.

quasi-isotropic material can be determined from the expressions

$$E^0 = \frac{E^*}{1 - {}^2/_5\,(\sigma_\alpha - \sigma_i)}, \quad G^0 = G^* \left(1 + \frac{2}{5}\frac{\sigma_\alpha - \sigma_i}{1 + \sigma_i}\right)^{-1};$$
$$\left(\sigma_\alpha = \frac{S_{13}^*}{S_{33}^*}, \quad \sigma_i = \frac{E^*}{2G^*} - 1\right). \tag{10}$$

Here, E^* and G^* are the Young's and shear moduli of the textured specimen, and σ_α and σ_i are the dispersion and isotropic Poisson's ratios, respectively.

Finally, expressions (7) make it possible to determine the anisotropy of the single crystal and calculate its elasticity constants. This is especially impotant for materials that have not yet been obtained in the form of single crystals. In order to determine these quantities it is necessary to find, apart from the means of elasticity constants of the quasi-isotropic material and the elasticity constants of the texture, the texture factor α. This can be done using the relation between α and the so-called texture coefficient C_4 introduced by Bunge [9]:

$$C_4 = -4\pi\, n_4 \alpha,$$

where $n_4 = -0.64636$ is the spherical function normalization factor for cubic symmetry. C_4 can also be calculated from the experimental x-ray diffraction curve for the investigated texture specimen [10, 11].

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14 January 1967 Krasnoyarsk